

Strong approximation of sets of finite perimeter in metric spaces *

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Abstract

In the setting of a metric space equipped with a doubling measure that supports a Poincaré inequality, we show that any set of finite perimeter can be approximated in the BV norm by a set whose topological and measure theoretic boundaries almost coincide. This result appears to be new even in the Euclidean setting. The work relies on a quasicontinuity-type result for BV functions proved by Lahti and Shanmugalingam (2016, [19]).

1 Introduction

It is well known in the Euclidean setting that a set of finite perimeter can be approximated in a weak sense by sets with smooth boundaries, see e.g. [3,

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Theorem 3.42]. In the setting of a much more general metric space, it was shown in [2] that a set of finite perimeter can be approximated in the L^1 -sense by sets whose boundaries are sufficiently regular that their Minkowski contents converge to the perimeter of the set.

On the other hand, fairly little seems to be known about approximating sets of finite perimeter in the BV *norm*. In the Euclidean setting, this type of result was given in [21, Theorem 3.1], where it was shown that given a set E of finite perimeter in an open set Ω and $\varepsilon > 0$, the set E can be approximated in the $\text{BV}(\Omega)$ -norm by a set F whose boundary $\partial F \cap \Omega$ is contained in a finite union of C^1 hypersurfaces, and so that $\mathcal{H}^{n-1}(\Omega \cap \partial F \setminus \partial^* F) < \varepsilon$, where $\partial^* F$ is the measure theoretic boundary.

In this paper we show a similar result in a metric space equipped with a doubling measure that supports a Poincaré inequality. More precisely, if $\Omega \subset X$ is an open set and $E \subset X$ is a set of finite perimeter in Ω , and $\varepsilon > 0$, we show that there exists a set $F \subset X$ with

$$\|\chi_F - \chi_E\|_{\text{BV}(\Omega)} < \varepsilon \quad \text{and} \quad \mathcal{H}(\Omega \cap \partial F \setminus \partial^* F) = 0,$$

where \mathcal{H} is the codimension 1 Hausdorff measure. This is given in Theorem 5.2. This is a partial generalization of [21, Theorem 3.1] to the metric setting, and in fact a partial improvement already in the Euclidean setting, since we are able to show that $\mathcal{H}(\Omega \cap \partial F \setminus \partial^* F)$ is zero instead of just being small. This is a fairly strong regularity requirement on the boundary, since in general the topological boundary of a set of finite perimeter can be much bigger than the measure theoretic boundary, see Example 5.3. The proof of Theorem 5.2 is heavily based on a quasicontinuity-type result for BV functions given in [19, Theorem 1.1].

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2 Notation and background

In this section we introduce the necessary notation and assumptions.

In this paper, (X, d, μ) is a complete metric space equipped with a Borel regular outer measure μ satisfying a doubling property, that is, there is a

constant $C_d \geq 1$ such that

$$0 < \mu(B(x, 2r)) \leq C_d \mu(B(x, r)) < \infty$$

for every ball $B = B(x, r)$ with center $x \in X$ and radius $r > 0$. Sometimes we abbreviate $\alpha B(x, r) := B(x, \alpha r)$, $\alpha > 0$. We assume that X consists of at least two points. By iterating the doubling condition, we obtain that for any $x \in X$ and $y \in B(x, R)$ with $0 < r \leq R < \infty$, we have

$$\frac{\mu(B(y, r))}{\mu(B(x, R))} \geq \frac{1}{C} \left(\frac{r}{R} \right)^Q, \quad (2.1)$$

where $C \geq 1$ and $Q > 0$ only depend on the doubling constant C_d . In general, $C \geq 1$ will denote a constant whose particular value is not important for the purposes of this paper, and might differ between each occurrence. When we want to specify that a constant C depends on the parameters a, b, \dots , we write $C = C(a, b, \dots)$. Unless otherwise specified, all constants only depend on the space X , more precisely on the doubling constant C_d , the constants C_P, λ associated with the Poincaré inequality defined below, and $\text{diam}(X)$.

A complete metric space with a doubling measure is proper, that is, closed and bounded sets are compact. Since X is proper, for any open set $\Omega \subset X$ we define $\text{Lip}_{\text{loc}}(\Omega)$ to be the space of functions that are Lipschitz in every $\Omega' \Subset \Omega$. Here $\Omega' \Subset \Omega$ means that Ω' is open and that $\overline{\Omega'}$ is a compact subset of Ω . Other local spaces of functions are defined similarly.

For any set $A \subset X$ and $0 < R < \infty$, the restricted spherical Hausdorff content of codimension 1 is defined by

$$\mathcal{H}_R(A) := \inf \left\{ \sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{r_i} : A \subset \bigcup_{i=1}^{\infty} B(x_i, r_i), r_i \leq R \right\}. \quad (2.2)$$

We define the above also for $R = \infty$ by requiring $r_i < \infty$. The codimension 1 Hausdorff measure of a set $A \subset X$ is given by

$$\mathcal{H}(A) := \lim_{R \rightarrow 0} \mathcal{H}_R(A).$$

For any outer measure ν on X , the codimension 1 Minkowski content of a set $A \subset X$ is defined by

$$\nu^+(A) := \liminf_{R \rightarrow 0} \frac{\nu \left(\bigcup_{x \in A} B(x, R) \right)}{2R}.$$

The measure theoretic boundary $\partial^* E$ of a set $E \subset X$ is the set of points $x \in X$ at which both E and its complement have positive upper density, i.e.

$$\limsup_{r \rightarrow 0} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} > 0 \quad \text{and} \quad \limsup_{r \rightarrow 0} \frac{\mu(B(x, r) \setminus E)}{\mu(B(x, r))} > 0.$$

The measure theoretic interior and exterior of E are defined respectively by

$$I_E := \left\{ x \in X : \lim_{r \rightarrow 0} \frac{\mu(B(x, r) \setminus E)}{\mu(B(x, r))} = 0 \right\} \quad (2.3)$$

and

$$O_E := \left\{ x \in X : \lim_{r \rightarrow 0} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} = 0 \right\}. \quad (2.4)$$

A curve γ is a rectifiable continuous mapping from a compact interval into X . A nonnegative Borel function g on X is an upper gradient of an extended real-valued function u on X if for all curves γ on X , we have

$$|u(x) - u(y)| \leq \int_{\gamma} g \, ds, \quad (2.5)$$

where x and y are the end points of γ . We interpret $|u(x) - u(y)| = \infty$ whenever at least one of $|u(x)|$, $|u(y)|$ is infinite. Of course, by replacing X with a set $A \subset X$ and considering curves γ in A , we can talk about a function g being an upper gradient of u in A . We define the local Lipschitz constant of a locally Lipschitz function $u \in \text{Lip}_{\text{loc}}(X)$ by

$$\text{Lip } u(x) := \limsup_{r \rightarrow 0} \sup_{y \in B(x, r) \setminus \{x\}} \frac{|u(y) - u(x)|}{d(y, x)}.$$

Then $\text{Lip } u$ is an upper gradient of u , see e.g. [8, Proposition 1.11]. Upper gradients were originally introduced in [13].

If g is a nonnegative μ -measurable function on X and (2.5) holds for 1-almost every curve, we say that g is a 1-weak upper gradient of u . A property holds for 1-almost every curve if it fails only for a curve family with zero 1-modulus. A family Γ of curves is of zero 1-modulus if there is a nonnegative Borel function $\rho \in L^1(X)$ such that for all curves $\gamma \in \Gamma$, the curve integral $\int_{\gamma} \rho \, ds$ is infinite.

Given an open set $\Omega \subset X$, we consider the following norm

$$\|u\|_{N^{1,1}(\Omega)} := \|u\|_{L^1(\Omega)} + \inf \|g\|_{L^1(\Omega)},$$

with the infimum taken over all 1-weak upper gradients g of u in Ω . The substitute for the Sobolev space $W^{1,1}(\Omega)$ in the metric setting is the Newton-Sobolev space

$$N^{1,1}(\Omega) := \{u : \|u\|_{N^{1,1}(\Omega)} < \infty\}.$$

It is known that for any $u \in N_{\text{loc}}^{1,1}(\Omega)$, there exists a minimal 1-weak upper gradient, denoted by g_u , that satisfies $g_u \leq g$ μ -almost everywhere in Ω , for any 1-weak upper gradient $g \in L_{\text{loc}}^1(\Omega)$ of u in Ω , see [5, Theorem 2.25]. For more on Newton-Sobolev spaces, we refer to [22, 5, 14].

Next we recall the definition and basic properties of functions of bounded variation on metric spaces, see [20]. See also e.g. [3, 9, 10, 23] for the classical theory in the Euclidean setting. For $u \in L_{\text{loc}}^1(X)$, we define the total variation of u in X to be

$$\|Du\|(X) := \inf \left\{ \liminf_{i \rightarrow \infty} \int_X g_{u_i} d\mu : u_i \in \text{Lip}_{\text{loc}}(X), u_i \rightarrow u \text{ in } L_{\text{loc}}^1(X) \right\},$$

where each g_{u_i} is an upper gradient of u_i . We say that a function $u \in L^1(X)$ is of bounded variation, and denote $u \in \text{BV}(X)$, if $\|Du\|(X) < \infty$. By replacing X with an open set $\Omega \subset X$ in the definition of the total variation, we can define $\|Du\|(\Omega)$. For an arbitrary set $A \subset X$, we define

$$\|Du\|(A) = \inf \{ \|Du\|(\Omega) : A \subset \Omega, \Omega \subset X \text{ is open} \}.$$

If $u \in \text{BV}(\Omega)$, $\|Du\|(\cdot)$ is a finite Radon measure on Ω by [20, Theorem 3.4]. The BV norm is defined by

$$\|u\|_{\text{BV}(\Omega)} := \|u\|_{L^1(\Omega)} + \|Du\|(\Omega).$$

A μ -measurable set $E \subset X$ is said to be of finite perimeter if $\|D\chi_E\|(X) < \infty$, where χ_E is the characteristic function of E . The perimeter of E in Ω is also denoted by

$$P(E, \Omega) := \|D\chi_E\|(\Omega).$$

Similarly as above, if $P(E, \Omega) < \infty$, then $P(E, \cdot)$ is a finite Radon measure on Ω . For any Borel sets $E_1, E_2 \subset X$ we have by [20, Proposition 4.7]

$$P(E_1 \cup E_2, X) \leq P(E_1, X) + P(E_2, X). \quad (2.6)$$

Similarly it can be shown that if $\Omega \subset X$ is an open set and $u, v \in L_{\text{loc}}^1(\Omega)$, then

$$\|D(u+v)\|(\Omega) \leq \|Du\|(\Omega) + \|Dv\|(\Omega). \quad (2.7)$$

We have the following coarea formula from [20, Proposition 4.2]: if $\Omega \subset X$ is an open set and $u \in L^1_{\text{loc}}(\Omega)$, then

$$\|Du\|(\Omega) = \int_{-\infty}^{\infty} P(\{u > t\}, \Omega) dt. \quad (2.8)$$

If $\|Du\|(\Omega) < \infty$, the above is true with Ω replaced by any Borel set $A \subset \Omega$.

We will assume throughout that X supports a $(1, 1)$ -Poincaré inequality, meaning that there exist constants $C_P \geq 1$ and $\lambda \geq 1$ such that for every ball $B(x, r)$, every $u \in L^1_{\text{loc}}(X)$, and every upper gradient g of u , we have

$$\int_{B(x, r)} |u - u_{B(x, r)}| d\mu \leq C_P r \int_{B(x, \lambda r)} g d\mu,$$

where

$$u_{B(x, r)} := \int_{B(x, r)} u d\mu := \frac{1}{\mu(B(x, r))} \int_{B(x, r)} u d\mu.$$

The 1-capacity of a set $A \subset X$ is given by

$$\text{Cap}_1(A) := \inf \|u\|_{N^{1,1}(X)},$$

where the infimum is taken over all functions $u \in N^{1,1}(X)$ such that $u \geq 1$ in A . For basic properties satisfied by the 1-capacity, such as monotonicity and countable subadditivity, see e.g. [5].

Given a set of finite perimeter $E \subset X$, for \mathcal{H} -almost every $x \in \partial^* E$ we have

$$\gamma \leq \liminf_{r \rightarrow 0} \frac{\mu(E \cap B(x, r))}{\mu(B(x, r))} \leq \limsup_{r \rightarrow 0} \frac{\mu(E \cap B(x, r))}{\mu(B(x, r))} \leq 1 - \gamma \quad (2.9)$$

where $\gamma \in (0, 1/2]$ only depends on the doubling constant and the constants in the Poincaré inequality, see [1, Theorem 5.4]. For an open set $\Omega \subset X$ and a μ -measurable set $E \subset X$ with $P(E, \Omega) < \infty$, we have for any Borel set $A \subset \Omega$

$$P(E, A) = \int_{\partial^* E \cap A} \theta_E d\mathcal{H}, \quad (2.10)$$

where $\theta_E: X \rightarrow [\alpha, C_d]$ with $\alpha = \alpha(C_d, C_P, \lambda) > 0$, see [1, Theorem 5.3] and [4, Theorem 4.6].

The lower and upper approximate limits of a μ -measurable function u on X are defined respectively by

$$u^\wedge(x) := \sup \left\{ t \in \overline{\mathbb{R}} : \lim_{r \rightarrow 0} \frac{\mu(B(x, r) \cap \{u < t\})}{\mu(B(x, r))} = 0 \right\} \quad (2.11)$$

and

$$u^\vee(x) := \inf \left\{ t \in \overline{\mathbb{R}} : \lim_{r \rightarrow 0} \frac{\mu(B(x, r) \cap \{u > t\})}{\mu(B(x, r))} = 0 \right\}. \quad (2.12)$$

Note that we understand BV functions to be μ -equivalence classes. To consider continuity properties, we need to consider the pointwise representatives u^\wedge and u^\vee . We also define the representative

$$\tilde{u} := (u^\wedge + u^\vee)/2. \quad (2.13)$$

3 Preliminary measure theoretic results

In this section we discuss some measure theoretic results that will be needed in the proof of our main result.

First we note that the following coarea inequality holds.

Lemma 3.1. *If $U \subset X$ is an open set and $w \in \text{Lip}_{\text{loc}}(U)$, then*

$$\int_{-\infty}^{\infty} \mathcal{H}(U \cap \partial\{w > t\}) dt \leq C_{co} \int_U \text{Lip } w d\mu,$$

where $C_{co} = C_{co}(C_d)$.

Proof. By [17, Proposition 3.5] (which is based on [7, Lemma 3.1]), the following coarea inequality holds: if ν is a positive Radon measure of finite mass and $u \in \text{Lip}(X)$ is bounded, then

$$\int_{-\infty}^{\infty} \nu^+(\partial\{u > t\}) dt \leq \int_X \text{Lip } u d\nu.$$

Choose $U'' \Subset U' \Subset U$ and let $\nu := \mu|_{U'}$, so that ν is of finite mass. Define a function $u := w$ in U' , so that $u \in \text{Lip}(U')$, and extend it to a bounded

function $u \in \text{Lip}(X)$. Since $\mathcal{H}(A) \leq C_d^3 \mu^+(A)$ for any $A \subset X$ (see e.g. [17, Proposition 3.12]), we have

$$\begin{aligned} \frac{1}{C_d^3} \int_{-\infty}^{\infty} \mathcal{H}(U'' \cap \partial\{w > t\}) dt &\leq \int_{-\infty}^{\infty} \mu^+(U'' \cap \partial\{w > t\}) dt \\ &= \int_{-\infty}^{\infty} \mu^+(U'' \cap \partial\{u > t\}) dt \\ &\leq \int_{-\infty}^{\infty} \nu^+(\partial\{u > t\}) dt \\ &\leq \int_X \text{Lip } u d\nu = \int_{U'} \text{Lip } w d\mu. \end{aligned}$$

By letting $U'' \nearrow U$ and using Lebesgue's monotone convergence theorem on both sides, we obtain the result with $C_{\text{co}} = C_d^3$. \square

Now we can show that the level sets of a locally Lipschitz function have the following weak regularity.

Proposition 3.2. *Let $U \subset X$ be an open set and let $w \in \text{Lip}_{\text{loc}}(U)$. Then $\mathcal{H}(U \cap \partial\{w > t\} \setminus \partial^*\{w > t\}) = 0$ for almost every $t \in \mathbb{R}$.*

Proof. Fix $U' \Subset U$. Note that $w \in \text{Lip}(U') \subset \text{BV}(U')$. Let

$$A := U' \cap \bigcup_{s \in \mathbb{R}} \partial^*\{w > s\}.$$

Note that in U , $\partial^*\{w > s\} \subset \partial\{w > s\} \subset \{w = s\}$, which are pairwise disjoint sets for distinct values of s . Note also that for any open set $V \subset U'$, denoting the minimal 1-weak upper gradient of w in V by g_w , we have

$$\int_V \text{Lip } w d\mu \leq C \int_V g_w d\mu \leq C \|Dw\|(V), \quad (3.1)$$

where the first inequality follows from the fact that $\text{Lip } w \leq C g_w$ μ -almost everywhere, see [8, Proposition 4.26] or [14, Proposition 13.5.2], and the second inequality follows from [12, Remark 4.7].

By using the disjointness of the sets $\partial^*\{u > s\} \subset \partial\{u > s\}$ in U , Lemma

3.1, and (3.1), we estimate for any open set V with $U' \setminus A \subset V \subset U'$

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathcal{H}(U' \cap \partial\{w > t\} \setminus \partial^*\{w > t\}) dt &= \int_{-\infty}^{\infty} \mathcal{H}(U' \cap \partial\{w > t\} \setminus A) dt \\
&\leq \int_{-\infty}^{\infty} \mathcal{H}(V \cap \partial\{w > t\}) dt \\
&\leq \int_V \text{Lip } w d\mu \\
&\leq C \|Dw\|(V).
\end{aligned}$$

By taking the infimum of open sets V as above, we get $C \|Dw\|(U' \setminus A)$ on the right-hand side. By also using the BV coarea inequality (2.8) and (2.10), we obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathcal{H}(U' \cap \partial\{w > t\} \setminus \partial^*\{w > t\}) dt &\leq C \|Dw\|(U' \setminus A) \\
&= C \int_{-\infty}^{\infty} P(\{w > t\}, U' \setminus A) dt \\
&\leq C \int_{-\infty}^{\infty} \mathcal{H}(U' \cap \partial^*\{w > t\} \setminus A) dt \\
&= 0
\end{aligned}$$

since $U' \cap \partial^*\{w > t\} \setminus A = \emptyset$ for all $t \in \mathbb{R}$. By exhausting U by sets $U' \Subset U$, we obtain the result. \square

Since we are going to work with quasicontinuity-type results, in the following we prove a few results on how to analyse and manipulate sets of small capacity.

Remark 3.3. The 1-capacity and the Hausdorff contents are closely related: it follows from [11, Theorem 4.3, Theorem 5.1] that $\text{Cap}_1(A) = 0$ if and only if $\mathcal{H}(A) = 0$. For any $R > 0$ and $A \subset X$, from the proof of [16, Lemma 3.4] it follows that

$$\text{Cap}_1(A) \leq C(C_d, C_P, \lambda, R) \mathcal{H}_R(A),$$

and by combining [11, Theorem 4.3] and the proof of [11, Theorem 5.1] we obtain that conversely

$$\mathcal{H}_R(A) \leq C(C_d, C_P, \lambda, R) \text{Cap}_1(A).$$

Finally, we note that Cap_1 is an outer capacity, meaning that

$$\text{Cap}_1(A) = \inf\{\text{Cap}_1(U) : U \supset A \text{ is open}\}$$

for any $A \subset X$, see e.g. [5, Theorem 5.31].

Lemma 3.4. *Let $A \subset X$ be a Borel set with $\mathcal{H}(A) < \infty$, and let $\varepsilon > 0$. Then there exists an open set $U \supset A$ with $\text{Cap}_1(U) \leq C\mathcal{H}(A) + \varepsilon$ such that*

$$r \frac{\mathcal{H}(A \cap B(x, r))}{\mu(B(x, r))} \rightarrow 0 \quad \text{as } r \rightarrow 0$$

uniformly for all $x \in X \setminus U$.

Proof. Inductively, we can pick compact sets $H_i \subset A \setminus \bigcup_{j=1}^{i-1} H_j$ with

$$\mathcal{H}\left(A \setminus \bigcup_{j=1}^i H_j\right) < 2^{-i}\varepsilon, \quad i \in \mathbb{N};$$

see e.g. [3, Proposition 1.43]. Also pick open sets $U_i \supset H_i$ and a decreasing sequence of numbers $1/5 \geq r_1 \geq r_2 \geq \dots$ with $\text{dist}(H_i, X \setminus U_i) \geq r_i$ such that

$$\text{Cap}_1(U_i) \leq \text{Cap}_1(H_i) + 2^{-i}\varepsilon \leq C\mathcal{H}_1(H_i) + 2^{-i}\varepsilon \leq C\mathcal{H}(H_i) + 2^{-i}\varepsilon;$$

see Remark 3.3.

Then define for each $i \in \mathbb{N}$

$$G_i := \left\{ x \in X \setminus \bigcup_{j=1}^i U_j : \exists r \in (0, r_i) \text{ such that } r \frac{\mathcal{H}(A \cap B(x, r))}{\mu(B(x, r))} \geq \frac{1}{i} \right\}.$$

Fix $i \in \mathbb{N}$. From the definition of G_i we obtain a covering $\{B(x, r(x))\}_{x \in G_i}$ of G_i , and by the 5-covering theorem, we can extract a countable collection of disjoint balls $\{B(x_k, r_k)\}_{k \in \mathbb{N}}$ such that the balls $B(x_k, 5r_k)$ cover G_i . Thus by Remark 3.3,

$$\begin{aligned} \text{Cap}_1(G_i) &\leq C\mathcal{H}_1(G_i) \leq C \sum_{k \in \mathbb{N}} \frac{\mu(B(x_k, 5r_k))}{5r_k} \leq C \sum_{k \in \mathbb{N}} \frac{\mu(B(x_k, r_k))}{r_k} \\ &\leq Ci \sum_{k \in \mathbb{N}} \mathcal{H}(A \cap B(x_k, r_k)) \leq Ci \mathcal{H}\left(A \setminus \bigcup_{j=1}^i H_j\right) \leq Ci 2^{-i}\varepsilon. \end{aligned}$$

Thus

$$\begin{aligned}
\text{Cap}_1 \left(\bigcup_{i \in \mathbb{N}} U_i \cup G_i \right) &\leq \sum_{i \in \mathbb{N}} \text{Cap}_1(U_i) + \sum_{i \in \mathbb{N}} \text{Cap}_1(G_i) \\
&\leq \sum_{i \in \mathbb{N}} (C\mathcal{H}(H_i) + 2^{-i}\varepsilon) + C \sum_{i \in \mathbb{N}} i 2^{-i} \varepsilon \\
&\leq C\mathcal{H}(A) + C\varepsilon.
\end{aligned}$$

For $x \in X \setminus \bigcup_{i \in \mathbb{N}} U_i \cup G_i$, if $0 < r < r_j$, then

$$r \frac{\mathcal{H}(A \cap B(x, r))}{\mu(B(x, r))} < \frac{1}{j}.$$

Finally, since

$$\text{Cap}_1 \left(A \setminus \bigcup_{i \in \mathbb{N}} U_i \right) \leq C\mathcal{H} \left(A \setminus \bigcup_{i \in \mathbb{N}} U_i \right) = 0,$$

we can choose an open set $V \supset A \setminus \bigcup_{i \in \mathbb{N}} U_i$ with $\text{Cap}_1(V) < \varepsilon$, and then we can take $U := \bigcup_{i \in \mathbb{N}} U_i \cup G_i \cup V$. \square

Lemma 3.5. *Let $G \subset X$ and $\varepsilon > 0$. Then there exists an open set $U \supset G$ with $\text{Cap}_1(U) \leq C \text{Cap}_1(G) + \varepsilon$ such that*

$$\frac{\mu(B(x, r) \cap G)}{\mu(B(x, r))} \rightarrow 0 \quad \text{as } r \rightarrow 0$$

uniformly for $x \in X \setminus U$.

Proof. We can assume that $\text{Cap}_1(G) < \infty$. By Remark 3.3, we have

$$\mathcal{H}_{\text{diam}(X)/10}(G) \leq C \text{Cap}_1(G).$$

(Of course we may have $\text{diam}(X)/10 = \infty$.) Thus we can pick a covering $\{B(x_k, r_k)\}_{k \in \mathbb{N}}$ of G with $r_k \leq \text{diam}(X)/10$ for all $k \in \mathbb{N}$ and

$$\sum_{k \in \mathbb{N}} \frac{\mu(B(x_k, r_k))}{r_k} \leq C \text{Cap}_1(G) + \varepsilon. \quad (3.2)$$

For any fixed $k \in \mathbb{N}$, consider the following three properties.

1. By [15, Lemma 6.2] (or more precisely its proof) we have

$$\frac{\mu(B(x_k, \rho))}{\rho} \leq CP(B(x_k, \rho), X)$$

for every $\rho \in [r_k, 2r_k]$; note that here we need the fact that $r_k \leq \text{diam}(X)/10$.

2. By applying the BV coarea formula (2.8) with $u(y) = \text{dist}(y, x_k)$ and $\Omega = B(x_k, 2r_k)$, we have $P(B(x_k, \rho), X) < \infty$ for almost every $\rho \in [r_k, 2r_k]$.
3. By applying the coarea inequality given in Lemma 3.1 with $w(y) = \text{dist}(y, x_k)$ and $U = B(x_k, 2r_k)$, we conclude that there exists $T \subset [r_k, 2r_k]$ with $\mathcal{L}^1(T) \geq r_k/2$ such that

$$\mathcal{H}(\partial B(x_k, \rho)) \leq 2C_{\text{co}} \frac{\mu(B(x_k, 2r_k))}{r_k} \leq 2C_{\text{co}} C_d \frac{\mu(B(x_k, \rho))}{\rho}$$

for every $\rho \in T$.

Thus for each $k \in \mathbb{N}$ we can find a radius $\tilde{r}_k \in [r_k, 2r_k]$ with

$$\begin{aligned} \mathcal{H}(\partial B(x_k, \tilde{r}_k)) &\leq C \frac{\mu(B(x_k, \tilde{r}_k))}{\tilde{r}_k} \\ &\leq CP(B(x_k, \tilde{r}_k), X) \leq C\mathcal{H}(\partial B(x_k, \tilde{r}_k)), \end{aligned} \tag{3.3}$$

where the last inequality follows from (2.10). Let $A := \bigcup_{k \in \mathbb{N}} \partial B(x_k, \tilde{r}_k)$, so that by the above and (3.2),

$$\begin{aligned} \mathcal{H}(A) &\leq \sum_{k \in \mathbb{N}} \mathcal{H}(\partial B(x_k, \tilde{r}_k)) \leq C \sum_{k \in \mathbb{N}} \frac{\mu(B(x_k, \tilde{r}_k))}{\tilde{r}_k} \\ &\leq C \sum_{k \in \mathbb{N}} \frac{\mu(B(x_k, r_k))}{r_k} \leq C \text{Cap}_1(G) + C\varepsilon. \end{aligned}$$

Note that if for any given ball $B(x, r)$ we have $\mathcal{H}(\partial B(x, r)) < \infty$, then for any $y \in X$ we have $\mathcal{H}(\partial B(x, r) \cap \partial B(y, s)) = 0$ for almost every $s > 0$. Thus we can pick the radii \tilde{r}_k recursively in such a way that we also have $\mathcal{H}(\partial B(x_k, \tilde{r}_k) \cap \partial B(x_l, \tilde{r}_l)) = 0$ whenever $k \neq l$.

Then take a set $U \supset A$ with

$$\text{Cap}_1(U) \leq C\mathcal{H}(A) + \varepsilon \leq C \text{Cap}_1(G) + C\varepsilon.$$

as given by Lemma 3.4. We can assume that also $U \supset \bigcup_{k \in \mathbb{N}} B(x_k, 2\tilde{r}_k)$, since by Remark 3.3 and (3.2),

$$\begin{aligned} \text{Cap}_1 \left(\bigcup_{k \in \mathbb{N}} B(x_k, 2\tilde{r}_k) \right) &\leq C\mathcal{H}_{\text{diam}(X)/5} \left(\bigcup_{k \in \mathbb{N}} B(x_k, 2\tilde{r}_k) \right) \\ &\leq C \sum_{k \in \mathbb{N}} \frac{\mu(B(x_k, 2\tilde{r}_k))}{2\tilde{r}_k} \\ &\leq C \sum_{k \in \mathbb{N}} \frac{\mu(B(x_k, r_k))}{r_k} \\ &\leq C \text{Cap}_1(G) + C\varepsilon. \end{aligned}$$

Let $x \in X \setminus U$. If for $r > 0$ we have $B(x_k, \tilde{r}_k) \cap B(x, r) \neq \emptyset$, then since $B(x_k, 2\tilde{r}_k) \subset U$, we have $\tilde{r}_k \leq \text{dist}(B(x_k, \tilde{r}_k), X \setminus U) \leq r$. Denoting $\tilde{B}_k := B(x_k, \tilde{r}_k)$, we have

$$\begin{aligned} \limsup_{r \rightarrow 0} \frac{\mu(B(x, r) \cap G)}{\mu(B(x, r))} &\leq \frac{\mu \left(B(x, r) \cap \bigcup_{k \in \mathbb{N}} \tilde{B}_k \right)}{\mu(B(x, r))} \\ &\leq r \frac{\sum_{\tilde{B}_k \cap B(x, r) \neq \emptyset} \mu(\tilde{B}_k)/\tilde{r}_k}{\mu(B(x, r))} \\ &\stackrel{(3.3)}{\leq} Cr \frac{\sum_{\tilde{B}_k \cap B(x, r) \neq \emptyset} \mathcal{H}(\partial \tilde{B}_k)}{\mu(B(x, r))} \\ &\leq Cr \frac{\mathcal{H}(A \cap B(x, 3r))}{\mu(B(x, r))} \\ &\rightarrow 0 \end{aligned}$$

uniformly as $r \rightarrow 0$ by Lemma 3.4. \square

The following lemma can be proved by very similar methods as those used above.

Lemma 3.6 ([18, Lemma 3.1]). *For any $G \subset X$, we can find an open set $U \supset G$ with $\text{Cap}_1(U) \leq C \text{Cap}_1(G)$ and $P(U, X) \leq C \text{Cap}_1(G)$.*

The next lemma gives a standard fact about the relationship between Hausdorff content and measure.

Lemma 3.7. *Let $A \subset X$ and $R > 0$. If $\mathcal{H}_R(A) = 0$, then $\mathcal{H}(A) = 0$.*

Note that the converse implication is trivial. In [16, Lemma 7.9] it was shown that we have the above even for $R = \infty$, under the additional assumption that the space is 1-hyperbolic, but we do not need to consider this assumption in this paper.

Proof. We can assume that A is bounded, and so $A \subset B(x_0, R_0)$ for some $x_0 \in X$ and $R_0 \geq R$. Fix $\varepsilon > 0$. By the fact that $\mathcal{H}_R(A) = 0$, we can find a covering $\{B(x_j, r_j)\}_{j \in \mathbb{N}}$ of A such that $r_j \leq R$ for all $j \in \mathbb{N}$ and

$$\sum_{j \in \mathbb{N}} \frac{\mu(B(x_j, r_j))}{r_j} < \varepsilon.$$

We can also assume that $B(x_j, r_j) \cap A \neq \emptyset$ for all $j \in \mathbb{N}$, and so $x_j \in B(x_0, 2R_0)$ for all $j \in \mathbb{N}$. Note that we can choose $Q > 1$ in (2.1). Then for each $j \in \mathbb{N}$ we have

$$\frac{\mu(B(x_0, 2R_0))}{(2R_0)^Q} r_j^{Q-1} \leq C \frac{\mu(B(x_j, r_j))}{r_j} < C\varepsilon,$$

so that

$$r_j < \left(C\varepsilon \frac{(2R_0)^Q}{\mu(B(x_0, 2R_0))} \right)^{1/(Q-1)} =: \delta_\varepsilon,$$

so in fact we have

$$\mathcal{H}_{\delta_\varepsilon}(A) \leq \sum_{j \in \mathbb{N}} \frac{\mu(B(x_j, r_j))}{r_j} < \varepsilon.$$

Here $\delta_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. Thus $\mathcal{H}(A) = \lim_{\varepsilon \rightarrow 0} \mathcal{H}_{\delta_\varepsilon}(A) = 0$. \square

The following lemma is well known e.g. in the Euclidean setting. We will only use it in the special case of sets of finite perimeter, but we give the standard proof for more general BV functions.

Lemma 3.8. *Let $\Omega \subset X$ be an open set, let $u \in L^1_{\text{loc}}(\Omega)$ with $\|Du\|(\Omega) < \infty$, and let $R > 0$. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $A \subset \Omega$ with $\mathcal{H}_R(A) < \delta$, then $\|Du\|(A) < \varepsilon$.*

Proof. By the BV coarea formula (2.8), for almost every $t \in \mathbb{R}$ we have $P(\{u > t\}, \Omega) < \infty$, and by (2.10) we have

$$\mathcal{H}(\partial^*\{u > t\} \cap \Omega) \leq CP(\{u > t\}, \Omega)$$

for such t . Fix one such $t \in \mathbb{R}$. Assume that there exists $\delta > 0$ and a sequence of Borel sets A_i , $i \in \mathbb{N}$, such that $\mathcal{H}_R(A_i) \leq 2^{-i}$ but $\mathcal{H}|_{\partial^*\{u > t\} \cap \Omega}(A_i) \geq \delta$. Then defining

$$A := \bigcap_{i \in \mathbb{N}} \bigcup_{j \geq i} A_j,$$

we have $\mathcal{H}_R(A) = 0$ but $\mathcal{H}(A) \geq \delta$, a contradiction by Lemma 3.7. Thus for almost every $t \in \mathbb{R}$, $\mathcal{H}|_{\partial^*\{u > t\} \cap \Omega}(A) \rightarrow 0$ if $\mathcal{H}_R(A) \rightarrow 0$, for A Borel.

By the coarea formula (2.8),

$$\|Du\|(A) = \int_{\mathbb{R}} P(\{u > t\}, A) dt$$

for any Borel set $A \subset \Omega$. Here we have by (2.10) again that $P(\{u > t\}, A) \leq C\mathcal{H}(\partial^*\{u > t\} \cap A)$ for almost every $t \in \mathbb{R}$. By using Lebesgue's dominated convergence theorem, with the majorant function $t \mapsto P(\{u > t\}, \Omega)$, we get $\|Du\|(A) \rightarrow 0$ if $\mathcal{H}_R(A) \rightarrow 0$, with A Borel. The result for general sets $A \subset \Omega$ follows by approximation. \square

Lemma 3.9. *Let $\Omega \subset X$ be an open set and let $u \in L^1_{\text{loc}}(\Omega)$ with $\|Du\|(\Omega) < \infty$. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $A \subset \Omega$ with $\text{Cap}_1(A) < \delta$, then $\|Du\|(A) < \varepsilon$.*

Proof. Combine Remark 3.3 and Lemma 3.8. \square

4 Quasicontinuity

In this section we present and slightly generalize the quasicontinuity-type result for BV functions given in [19].

In the Euclidean setting, results on the fine properties of BV functions can be formulated in terms of the lower and upper approximate limits u^\wedge and u^\vee given in (2.11) and (2.12). In the metric setting, we need to consider more than two jump values. Recall the definition of the number γ from (2.9).

Then we define the functions u^l , $l = 1, \dots, n := \lfloor 1/\gamma \rfloor$, as follows: $u^1 := u^\wedge$, $u^n := u^\vee$, and for $l = 2, \dots, n-1$ we define inductively

$$u^l(x) := \sup \left\{ t \in \overline{\mathbb{R}} : \lim_{r \rightarrow 0} \frac{\mu(B(x, r) \cap \{u^{l-1}(x) + \delta < u < t\})}{\mu(B(x, r))} = 0 \quad \forall \delta > 0 \right\} \quad (4.1)$$

provided $u^{l-1}(x) < u^\vee(x)$, and otherwise we set $u^l(x) = u^\vee(x)$. It can be shown that each u^l is a Borel function, and $u^\wedge = u^1 \leq \dots \leq u^n = u^\vee$.

We have the following notion of quasicontinuity for BV functions.

Theorem 4.1 ([19, Theorem 1.1]). *Let $u \in \text{BV}(X)$ and let $\varepsilon > 0$. Then there exists an open set $G \subset X$ with $\text{Cap}_1(G) < \varepsilon$ such that if $y_k \rightarrow x$ with $y_k, x \in X \setminus G$, then*

$$\min_{l_2 \in \{1, \dots, n\}} |u^{l_1}(y_k) - u^{l_2}(x)| \rightarrow 0$$

for each $l_1 = 1, \dots, n$.

First we give a local version of this result, as follows.

Corollary 4.2. *Let $\Omega \subset X$ be an open set, let $u \in \text{BV}_{\text{loc}}(\Omega)$, and let $\varepsilon > 0$. Then there exists an open set $G \subset \Omega$ with $\text{Cap}_1(G) < \varepsilon$ such that if $y_k \rightarrow x$ with $y_k, x \in \Omega \setminus G$, then*

$$\min_{l_2 \in \{1, \dots, n\}} |u^{l_1}(y_k) - u^{l_2}(x)| \rightarrow 0$$

for each $l_1 = 1, \dots, n$.

Proof. Pick sets $\Omega_1 \Subset \Omega_2 \Subset \dots$ with $\Omega = \bigcup_{j \in \mathbb{N}} \Omega_j$. Also pick cutoff functions $\eta_j \in \text{Lip}_c(\Omega_{j+1})$ with $0 \leq \eta_j \leq 1$ and $\eta_j = 1$ in Ω_j for each $j \in \mathbb{N}$. Denote the Lipschitz constants by L_j . Fix $j \in \mathbb{N}$. We have $u \in \text{BV}(\Omega_{j+1})$, so that we find a sequence $\text{Lip}_{\text{loc}}(\Omega_{j+1}) \ni u_i \rightarrow u$ in $L^1_{\text{loc}}(\Omega_{j+1})$ with

$$\lim_{i \rightarrow \infty} \int_{\Omega_{j+1}} g_{u_i} d\mu = \|Du\|(\Omega_{j+1}).$$

Recall that g_{u_i} denotes the minimal 1-weak upper gradient of u_i . Clearly $\eta_j u_i \in \text{Lip}(X)$ with $\eta_j u_i \rightarrow \eta_j u$ in $L^1(X)$ as $i \rightarrow \infty$. Thus by the definition

of the total variation and by the Leibniz rule for Newton-Sobolev functions, see [5, Theorem 2.15], we have

$$\begin{aligned}
\|D(\eta_j u)\|(X) &\leq \liminf_{i \rightarrow \infty} \int_X g_{\eta_j u_i} d\mu \\
&\leq \liminf_{i \rightarrow \infty} \int_X g_{\eta_j} |u_i| + \eta_j g_{u_i} d\mu \\
&\leq \limsup_{i \rightarrow \infty} \int_{\text{supp}(\eta_j)} L_j |u_i| d\mu + \limsup_{i \rightarrow \infty} \int_{\Omega_{j+1}} g_{u_i} d\mu \\
&\leq L_j \|u\|_{L^1(\Omega_{j+1})} + \|Du\|(\Omega_{j+1}) < \infty.
\end{aligned}$$

Thus $u_j := \eta_j u \in \text{BV}(X)$ for each $j \in \mathbb{N}$, and so we can apply Theorem 4.1 to obtain open sets $G_j \subset X$ with $\text{Cap}_1(G_j) < 2^{-j}\varepsilon$. Defining $G := \bigcup_{j \in \mathbb{N}} G_j \cap \Omega$, we have $\text{Cap}_1(G) < \varepsilon$, and if $y_k \rightarrow x$ with $y_k, x \in \Omega \setminus G$, then $y_k, x \in \Omega_j$ for some $j \in \mathbb{N}$ and thus for large enough $k \in \mathbb{N}$

$$\min_{l_2 \in \{1, \dots, n\}} |u^{l_1}(y_k) - u^{l_2}(x)| = \min_{l_2 \in \{1, \dots, n\}} |(u\eta_j)^{l_1}(y_k) - (u\eta_j)^{l_2}(x)| \rightarrow 0$$

as $k \rightarrow \infty$ for each $l_1 = 1, \dots, n$, by the fact that $y_k, x \notin G_j$. □

Recall the definitions of the measure theoretic interior and exterior I_E and O_E of a set $E \subset X$ from (2.3) and (2.4). Note that for $u = \chi_E$, we have $x \in I_E$ if and only if $u^\wedge(x) = u^\vee(x) = 1$, $x \in O_E$ if and only if $u^\wedge(x) = u^\vee(x) = 0$, and $x \in \partial^* E$ if and only if $u^\wedge(x) = 0$ and $u^\vee(x) = 1$. Moreover, in this case $u^1 = u^\wedge$ and $u^2 = \dots = u^n = u^\vee$.

In this paper we will only need the following notion of quasicontinuity for sets of finite perimeter, which is obtained by applying Corollary 4.2 to $u = \chi_E$.

Corollary 4.3. *Let $\Omega \subset X$ be an open set, let $E \subset X$ be a μ -measurable set with $P(E, \Omega) < \infty$, and let $\varepsilon > 0$. Then there exists an open set $G \subset \Omega$ with $\text{Cap}_1(G) < \varepsilon$ such that if $y_k \rightarrow x$ with $y_k, x \in \Omega \setminus G$, then*

$$\min\{|\chi_E^\wedge(y_k) - \chi_E^\wedge(x)|, |\chi_E^\wedge(y_k) - \chi_E^\vee(x)|\} \rightarrow 0$$

and

$$\min\{|\chi_E^\vee(y_k) - \chi_E^\wedge(x)|, |\chi_E^\vee(y_k) - \chi_E^\vee(x)|\} \rightarrow 0.$$

For example, if $x \in O_E$, then $\chi_E^\wedge(x) = 0 = \chi_E^\vee(x)$ and necessarily $y_k \in O_E$ for sufficiently large $k \in \mathbb{N}$.

5 Approximation of sets of finite perimeter

In this section we prove our main result on the approximation of a set of finite perimeter by more regular sets in the BV norm.

We will need to work with Whitney-type coverings of open sets. For the construction of such coverings and their properties, see e.g. [6, Theorem 3.1]. Given any open set $U \subset X$ and a scale $R > 0$, we can choose a Whitney-type covering $\{B_j = B(x_j, r_j)\}_{j=1}^\infty$ of U such that

1. for each $j \in \mathbb{N}$,

$$r_j = \min \left\{ \frac{\text{dist}(x_j, X \setminus U)}{40\lambda}, R \right\}, \quad (5.1)$$

2. for each $k \in \mathbb{N}$, the ball $10\lambda B_k$ meets at most $C = C(C_d, \lambda)$ balls $10\lambda B_j$ (that is, a bounded overlap property holds),
3. if $10\lambda B_j$ meets $10\lambda B_k$, then $r_j \leq 2r_k$.

Given such a covering of U , we can take a partition of unity $\{\phi_j\}_{j=1}^\infty$ subordinate to the covering, such that $0 \leq \phi_j \leq 1$, each ϕ_j is a C/r_j -Lipschitz function, and $\text{supp}(\phi_j) \subset 2B_j$ for each $j \in \mathbb{N}$ (see e.g. [6, Theorem 3.4]). Finally, we can define a *discrete convolution* v of any $u \in L^1_{\text{loc}}(U)$ with respect to the Whitney-type covering by

$$v := \sum_{j=1}^\infty u_{B_j} \phi_j. \quad (5.2)$$

In general, v is locally Lipschitz in U , and hence belongs to $L^1_{\text{loc}}(U)$.

We can “mollify” BV functions in open sets in the following manner. Recall the definition of the pointwise representative \tilde{u} from (2.13).

Theorem 5.1. *Let $U \subset \Omega \subset X$ be open sets, and let $u \in L^1_{\text{loc}}(X)$ with $\|Du\|(\Omega) < \infty$. Then there exists a function $w \in L^1_{\text{loc}}(X)$ with $\|Dw\|(\Omega) < \infty$ such that $w = u$ in $\Omega \setminus U$ and $\tilde{w}|_U \in N^{1,1}(U) \cap \text{Lip}_{\text{loc}}(U)$ with an upper gradient g satisfying $\|g\|_{L^1(U)} \leq C\|Du\|(U)$.*

The function w is defined in U as a limit of discrete convolutions of u with respect to Whitney-type coverings of open sets $U_1 \subset U_2 \subset \dots \subset U$ with

$U = \bigcup_{i \in \mathbb{N}} U_i$, at an arbitrary fixed scale $R > 0$. For \mathcal{H} -almost every $x \in \partial U$ we have

$$\frac{1}{\mu(B(x, r))} \int_{B(x, r) \cap U} |w - u| d\mu \rightarrow 0 \quad (5.3)$$

as $r \rightarrow 0$.

This is essentially [19, Corollary 3.6]. The last two sentences of the theorem are not part of [19, Corollary 3.6], but follow from its proof. Moreover, in [19, Corollary 3.6] we make the assumption $u \in \text{BV}(\Omega)$, but the proof runs through almost verbatim for the slightly more general case presented here.

Now we give our main result.

Theorem 5.2. *Let $\Omega \subset X$ be an open set, let $E \subset X$ be a μ -measurable set with $P(E, \Omega) < \infty$, and let $\varepsilon > 0$. Then there exists a μ -measurable set $F \subset X$ with*

$$\|\chi_F - \chi_E\|_{\text{BV}(\Omega)} < \varepsilon \quad \text{and} \quad \mathcal{H}(\Omega \cap \partial F \setminus \partial^* F) = 0.$$

Proof. Apply Corollary 4.3 to obtain a set $G \subset \Omega$ with $\text{Cap}_1(G) < \varepsilon$, and then apply Lemma 3.5 to obtain an open set $U \subset \Omega$ with $U \supset G$ such that

$$\frac{\mu(B(x, r) \cap G)}{\mu(B(x, r))} \rightarrow 0 \quad \text{as } r \rightarrow 0$$

uniformly for $x \in \Omega \setminus U$. By Lemma 3.9 we can also assume that

$$\|D\chi_E\|(U) < \varepsilon. \quad (5.4)$$

In the following, we “mollify” χ_E in the set U and then define F as a super-level set of the mollified function. First, apply Theorem 5.1 with $u = \chi_E$ and at the scale $R = 1$ to obtain a function $w \in L^1_{\text{loc}}(X)$ with $\|Dw\|(\Omega) < \infty$ and $\tilde{w} \in \text{Lip}_{\text{loc}}(U)$.

Fix $x \in \Omega \cap \partial U$ with $x \in O_E$. By Corollary 4.3, there exists $\delta \in (0, 1)$ such that $B(x, \delta) \subset \Omega$ and

$$y \in O_E \quad \text{for all } y \in B(x, \delta) \setminus G. \quad (5.5)$$

By making δ smaller, if necessary, by Lemma 3.5 we also have

$$\frac{\mu(B(z, r) \cap G)}{\mu(B(z, r))} < \frac{1}{4C_d^{\lceil \log_2(200\lambda) \rceil}} \quad (5.6)$$

for all $z \in X \setminus U$ and $r \in (0, \delta)$. Here $\lceil a \rceil$ is the smallest integer at least $a \in \mathbb{R}$.

Fix $y \in B(x, \delta/4) \cap U$. Recall that w is defined in U as a limit of discrete convolutions of u with respect to Whitney-type coverings $\{B_j^i = B(x_j^i, r_j^i)\}_{j \in \mathbb{N}}$ of sets $U_i \subset U$ at scale $R = 1$. Since $U = \bigcup_{i \in \mathbb{N}} U_i$, we can fix a sufficiently large $i \in \mathbb{N}$ such that

$$\text{dist}(y, X \setminus U_i) \geq \text{dist}(y, X \setminus U)/2.$$

Suppose that $y \in B(x_j^i, 2r_j^i)$. It is easy to see that $B(x_j^i, 2r_j^i) \subset B(x, \delta)$. Then

$$\begin{aligned} r_j^i &= \min \left\{ \frac{\text{dist}(x_j^i, X \setminus U_i)}{40\lambda}, R \right\} = \frac{\text{dist}(x_j^i, X \setminus U_i)}{40\lambda} \\ &\geq \frac{\text{dist}(y, X \setminus U_i) - 2r_j^i}{40\lambda} \\ &\geq \frac{\text{dist}(y, X \setminus U)}{80\lambda} - \frac{r_j^i}{20\lambda}. \end{aligned}$$

Thus

$$r_j^i \geq \frac{\text{dist}(y, X \setminus U)}{90\lambda}.$$

Since $B(x, \delta) \subset \Omega$, there is $z \in \Omega \setminus U$ with $d(y, z) = \text{dist}(y, X \setminus U)$. Then $B(z, 2d(y, z)) \subset 200\lambda B_j^i$, so by the doubling property of the measure

$$\mu(B(z, 2d(y, z))) \leq C_d^{\lceil \log_2(200\lambda) \rceil} \mu(B_j^i).$$

Moreover,

$$d(y, z) = \text{dist}(y, X \setminus U) \leq d(y, x) < \delta/4,$$

and thus $d(x, z) < \delta/2$. Hence $B(z, 2d(y, z)) \subset B(x, \delta)$, so that

$$2B_j^i \setminus O_E \subset B(z, 2d(y, z)) \setminus O_E \subset B(z, 2d(y, z)) \cap G$$

by (5.5). Using this and (5.6), we obtain

$$\begin{aligned} u_{B_j^i} &= \frac{\mu(E \cap B_j^i)}{\mu(B_j^i)} \leq \frac{C_d^{\lceil \log_2(200\lambda) \rceil}}{\mu(B(z, 2d(y, z)))} \mu(B(z, 2d(y, z)) \cap E) \\ &\leq \frac{C_d^{\lceil \log_2(200\lambda) \rceil}}{\mu(B(z, 2d(y, z)))} \mu(B(z, 2d(y, z)) \cap G) \\ &\leq \frac{1}{4}. \end{aligned}$$

For each $i \in \mathbb{N}$, let w_i be the discrete convolution of u in U_i with respect to the Whitney-type covering $\{B_j^i\}_{j \in \mathbb{N}}$. Recalling the definition of a discrete convolution from (5.2), we have for suitable Lipschitz functions ϕ_j^i

$$w_i(y) = \sum_{j \in \mathbb{N}} u_{B_j^i} \phi_j^i(y) \leq \frac{1}{4} \sum_{j \in \mathbb{N}} \phi_j^i(y) = \frac{1}{4}.$$

According to Theorem 5.1, the quantity $\tilde{w}(y)$ is defined as the limit of $w_i(y)$ as $i \rightarrow \infty$, so we have $\tilde{w}(y) \leq 1/4$. Since $y \in B(x, \delta/4) \cap U$ was arbitrary, we have $\tilde{w} \leq 1/4$ in $B(x, \delta/4) \cap U$. Similarly, for any $x \in \Omega \cap \partial U \cap I_E$ there exists some $r > 0$ such that $\tilde{w} \geq 3/4$ in $B(x, r) \cap U$.

By the BV coarea formula (2.8), we can find a set $T \subset (1/4, 3/4)$ with $\mathcal{L}^1(T) \geq 1/4$ such that for all $t \in T$,

$$\|D\chi_{\{w>t\}}\|(U) \leq 4\|Dw\|(U) \leq C\|D\chi_E\|(U), \quad (5.7)$$

where the last inequality follows from Theorem 5.1. By (5.3), there exists $N \subset \partial U$ with $\mathcal{H}(N) = 0$ such that for every $x \in \partial U \setminus N$, we have

$$\frac{1}{\mu(B(x, r))} \int_{B(x, r) \cap U} |w - \chi_E| d\mu \rightarrow 0$$

as $r \rightarrow 0$. For any fixed $t \in (0, 1)$, this implies

$$\begin{aligned} & \frac{1}{\mu(B(x, r))} \int_{B(x, r) \cap U} |\chi_{\{w>t\}} - \chi_E| d\mu \\ & \leq \frac{1}{\min\{t, 1-t\}} \frac{1}{\mu(B(x, r))} \int_{B(x, r) \cap U} |w - \chi_E| d\mu \rightarrow 0 \end{aligned} \quad (5.8)$$

as $r \rightarrow 0$.

Again by the BV coarea formula (2.8), for almost every $t \in (0, 1)$, setting

$$F_t := (I_E \cap \Omega \setminus U) \cup (\{\tilde{w} > t\} \cap U),$$

so that $F_t = \{w > t\} \cap \Omega$ as μ -equivalence classes, we have $P(F_t, \Omega) < \infty$. By (5.8), for every $x \in \Omega \setminus (U \cup N)$ and for all $s \neq 0$, we have $x \notin \partial^* \{\chi_{F_t} - \chi_E > s\}$. Thus by the BV coarea formula (2.8) and (2.10), for almost every

$t \in (0, 1)$

$$\begin{aligned}
\|D(\chi_{F_t} - \chi_E)\|(\Omega \setminus U) &= \int_{-\infty}^{\infty} P(\{\chi_{F_t} - \chi_E > s\}, \Omega \setminus U) ds \\
&\leq C \int_{-\infty}^{\infty} \mathcal{H}(\partial^* \{\chi_{F_t} - \chi_E > s\} \cap (\Omega \setminus U)) ds \\
&= 0.
\end{aligned}$$

By using this and (5.7), we have for almost every $t \in T$

$$\begin{aligned}
\|D(\chi_{F_t} - \chi_E)\|(\Omega) &= \|D(\chi_{F_t} - \chi_E)\|(U) \\
&\stackrel{(2.7)}{\leq} \|D\chi_{F_t}\|(U) + \|D\chi_E\|(U) \\
&\leq C\|D\chi_E\|(U) + \|D\chi_E\|(U) \\
&< C\varepsilon
\end{aligned}$$

by (5.4), and also

$$\mathcal{H}(U \cap \partial\{\tilde{w} > t\} \setminus \partial^*\{\tilde{w} > t\}) = 0 \quad (5.9)$$

by Proposition 3.2. We fix one such t and define $F := F_t$. Since

$$\|\chi_F - \chi_E\|_{L^1(\Omega)} \leq \mu(U) \leq \text{Cap}_1(U) < \varepsilon,$$

we have $\|\chi_F - \chi_E\|_{\text{BV}(\Omega)} < C\varepsilon$ and one claim of the theorem is proved.

From Corollary 4.3 we know that if $x \in \partial I_E \cap \Omega \setminus \overline{U}$, then $x \in \partial^* E$. Thus from the definition of F it follows that

$$\partial F \cap \Omega \setminus \overline{U} = \partial I_E \cap \Omega \setminus \overline{U} = \partial^* E \cap \Omega \setminus \overline{U} = \partial^* F \cap \Omega \setminus \overline{U}.$$

If $x \in \Omega \cap \partial U \cap O_E$, the previously proved fact that $\tilde{w} \leq 1/4$ in $B(x, r) \cap U$ for some $r > 0$ implies that

$$\chi_{\{\tilde{w} > t\}}(y) = 0 \quad \text{for all } y \in B(x, r) \cap U$$

for any $t \in (1/4, 3/4)$. Combining this with (5.5), we conclude that x is an exterior point of F . Analogously, if $x \in \Omega \cap \partial U \cap I_E$, then x is an interior point of F . If $x \in \Omega \cap \partial U \cap \partial^* E \setminus N$, then $x \in \partial U \cap \partial^* F$ by (5.8). In total,

$$\partial F \setminus \partial^* F \subset (U \cap \partial F \setminus \partial^* F) \cup N = (U \cap \partial\{\tilde{w} > t\} \setminus \partial^*\{\tilde{w} > t\}) \cup N.$$

Hence

$$\mathcal{H}(\partial F \setminus \partial^* F) \leq \mathcal{H}(U \cap \partial\{\tilde{w} > t\} \setminus \partial^*\{\tilde{w} > t\}) = 0$$

by (5.9). □

Example 5.3. A standard example illustrating how badly behaved a set of finite perimeter can be is given by the so-called *enlarged rationals*. Consider the Euclidean space \mathbb{R}^2 equipped with the Lebesgue measure \mathcal{L}^2 . Let $\{q_i\}_{i \in \mathbb{N}}$ be an enumeration of $\mathbb{Q} \times \mathbb{Q} \subset \mathbb{R}^2$, and define

$$E := \bigcup_{i \in \mathbb{N}} B(q_i, 2^{-i}).$$

Clearly $\mathcal{L}^2(E) \leq \pi$. By the lower semicontinuity and subadditivity of perimeter, see (2.6), we can estimate

$$P(E, \mathbb{R}^2) \leq \sum_{i=1}^{\infty} P(B(q_i, 2^{-i}), \mathbb{R}^2) \leq 2\pi \sum_{i=1}^{\infty} 2^{-i},$$

so that $P(E, \mathbb{R}^2) < \infty$, and then also $\mathcal{H}(\partial^* E) < \infty$. On the other hand, $\partial E = \mathbb{R}^2 \setminus E$, so that $\mathcal{L}^2(\partial E) = \infty$ and in particular $\mathcal{H}^1(\partial E) = \infty = \mathcal{H}(\partial E)$ (where \mathcal{H}^1 is the 1-dimensional Hausdorff measure, which is comparable to the codimension 1 Hausdorff measure \mathcal{H}). However, we can define the set $F \subset \mathbb{R}^2$ of Theorem 5.2 as

$$F := \bigcup_{i=1}^N B(q_i, 2^{-i})$$

for $N \in \mathbb{N}$ sufficiently large. It can then be shown that $\|\chi_F - \chi_E\|_{\text{BV}(\mathbb{R}^2)} \rightarrow 0$ as $N \rightarrow \infty$, and that $\mathcal{H}(\partial F \setminus \partial^* F) = 0$. By slightly modifying the set F near the intersections of the spheres $\partial B(q_i, 2^{-i})$, if necessary, we can even ensure that $\partial F = \partial^* F$.

Open Problem. In Theorem 5.2, is it possible to obtain $\partial F \cap \Omega = \partial^* F \cap \Omega$?

If the answer is yes, note that $\text{int}(F) \cap \Omega = I_F \cap \Omega$, and thus in Ω , $\chi_F^\wedge = \chi_{I_F}$ is a lower semicontinuous function. Similarly, in Ω , $\chi_F^\vee = \chi_{I_F \cup \partial^* F} = \chi_{\overline{F}}$ is then an upper semicontinuous function.

Note also that it follows from the proof of Theorem 5.2 that χ_F^\wedge and χ_E^\wedge can differ only in the set $U \cup N$, where N is the \mathcal{H} -negligible set defined before (5.8). Thus we have

$$\text{Cap}_1(\{\chi_F^\wedge \neq \chi_E^\wedge\}) < \varepsilon \quad \text{and similarly} \quad \text{Cap}_1(\{\chi_F^\vee \neq \chi_E^\vee\}) < \varepsilon.$$

For a more general BV function, we can now ask the following.

Open Problem. Let $\Omega \subset X$ be an open set, let $u \in \text{BV}_{\text{loc}}(\Omega)$, and let $\varepsilon > 0$. Can we find a function $v \in \text{BV}_{\text{loc}}(\Omega)$ with $\|v - u\|_{\text{BV}(\Omega)} < \varepsilon$,

$$\text{Cap}_1(\{v^\wedge \neq u^\wedge\}) < \varepsilon, \quad \text{Cap}_1(\{v^\vee \neq u^\vee\}) < \varepsilon,$$

and such that v^\wedge is lower semicontinuous and v^\vee is upper semicontinuous?

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